A CLASSICAL AND QUANTUM MECHANICAL ANALOG OF TWO CAPACITORS PARADOX

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Abstract

As it is well-known one of the most fascinating examples in remarkable discussion between Einstein and Bohr on the conceptual foundation of the quantum mechanics (Heisenberg energy-time uncertainty relation especially) was an experimental device representing a box hanged on an elastic spring. The pair of similar devices is used in this work for formulation of a classical and (implicitly) quantum mechanical analog of the famous two capacitors paradox. It admits a simple solution of the paradox since energy difference or seeming paradoxical "loss" can be explained by work of the elastic force for moving of the boxes in the gravitational field. (Obviously, original two capacitors paradox can be explained in the analogous way.)

As it is well-known remarkable two-capacitors paradox, formulated and considered in many textbooks and articles on the basic principles and applications of the electronic and electrodynamics [1]-[7], states the following. Consider an ideal (without any electrical resistance and inductivity) electrical circuit with first, initially charged, and second, initially non-charged, of two identical capacitors. In given circuit, by transition from initial, open state (switch OFF state) in the closed state (switch ON state), an unexpected, mysterious loss of the half of initial energy of electrical fields within capacitors occurs. Different authors [4]-[7] suggest that given energy loss is realized by different dissipative processes (Joule heating or/and electromagnetic waves emissions) realized by non-neglectable residual electric resistances and inductivities in realistic circuits.

As it is well-known one of the most fascinating examples in remarkable discussion between Einstein and Bohr on the conceptual foundation of the quantum mechanics (Heisenberg energy-time uncertainty relation especially) [8] was an experimental device representing a box hanged on an elastic spring. The pair of similar devices will be used in this work for formulation of a classical and (implicitly) quantum mechanical analog of the two capacitor paradox. It admits a simple solution of the paradox since energy difference or seeming paradoxical "loss" can be explained by work of the elastic force for moving of the box in gravitational field. (Obviously, original two capacitors paradox can be explained in the analogous way.)

In remarkable Einstein-Bohr discussion on the conceptual problems of the quantum mechanics foundation (Heisenberg coordinate-momentum and energy-time uncertainty relation especially) [8] the following experimental device was especially interesting. It represents a box, for example a cubic box, hanged on a elastic spring in Earth gravitational field (practically constant nearly Earth surface) so that elastic and gravitational force are initially in enquilibrium. Given box holds an external pointer that points out position of the box, i.e. equilibrium point, on a vertical length scale fixed without box.

At the center of a box vertical side there is a small hole that can be closed either open by corresponding mechanism connected with a clock. When mechanism opens hole in a time moment determined by clock single photon can leave the box. After photon leave of the box mass of the box, according to equivalence principle, becomes smaller and elastic force becomes larger than gravitational. For this reason elastic force lift the box toward a higher point on the scale representing new equilibrium point. Given lifting can be considered as the work done by elastic force.

Now, we shall consider the pair of similar experimental devices for formulation of a classical and quantum mechanical analog of remarkable two capacitor paradox [1]-[7].

As well as in the mentioned Einstein-Bohr discussion we shall use first box spring replelte completely by a liquid. Suppose that total mass of the liquid initially equals M. Then equilibrium condition between gravitational and elastic force $Mg = kX$, where k represents spring elasticity coefficient and X - box position, determines this position by expression

$$X = \frac{Mg}{k}. \tag{1}$$

Energy of the elastic force in this position equals, as it is well-known,

$$E_{in1} = \frac{kX^2}{2} = \frac{M^2g^2}{2k} \tag{2}$$

After opening of the hole at vertical side of the box by mentioned mechanism the following occurs. Through hole, in an admitable approximation, discretely, drop by drop any of which holds mass $m = \frac{M}{N}$ for $N \gg 1$, there is a free fall of the fluid drops in the second, initially empty, neighbouring box, placed immediately under the first box. This second box holds form identical to the first, but it does not hold high horizontal side so that free falling drops can arrive inside the second box. Suppose, also, that given second box is placed at a vertical spring as well as that this box holds a pointer which points out position of the second box, i.e. equilibrium point between gravitational force acting at liquid and elastic force. Since second box is initially empty initial energy of corresponding elastic force is zero.

In this way initial total energy of both elastic forces, elastic force acting at the first box and elastic force acting at the second box initially, equals

$$E_{in} = E_{in1} + 0 = E_{in1} \tag{3}$$

that is identical to E_{in1} (2).

As it is not hard to see liquid will turn from the first in the second box till final moment when masses in both boxes become equivalent and equal $\frac{M}{2}$. In this moment energies of the elastic force acting on the first and elastic force acting on the second box will be equivalent and will equal

$$E_{fin1,2} = (\frac{M}{2})^2 \frac{g^2}{2k} = \frac{1}{4} M^2 \frac{g^2}{2k} = \frac{1}{4} E_{in1}. \tag{4}$$

Then total energy of both elastic forces equals

$$E_{fin} = 2E_{fin1,2} = \frac{1}{2}E_{in1} = \frac{1}{2}E_{in}. \tag{5}$$

Obviously, final total energy of the elastic forces is two times smaller that initial total energy of the elastic forces and we have a (seemingly) paradoxical energy loss equivalent to one half of given initial energy. This is, of course, a complete analogy with two capacitor paradox [1]-[7].

For explanation of given seeming paradox consider dynamics of the box and liquid, i.e. drops more detailedly.

Suppose firstly that distance between boxes are small and that kinetic energy that drop obtains by gravitational force by free faling between two boxes can be neglected.

After free falling of the first drop in the initially empty second box mass increases for m till m. For this reason appears gravitational force mg larger than zero elastic force of the spring. It causes compression of the spring, i.e. moving of the box down from 0 for $q = \frac{mg}{k}$ till q representing new equilibrium position. It corresponds to increase of the spring elastic force energy from 0 for $\frac{kq^2}{2}$ till $\frac{kq^2}{2}$.

By simple induction we can conclude the following. After free falling of the n-th drop in the second box with mass $(n-1)m$ before drop falling, mass of the box increases for m till nm. For this reason new gravitational force nmg becomes larger than elastic force of the spring $(n-1)kq$. It causes further compression of the spring, i.e. moving of the box down from $(n-1)q$ for $q = \frac{mg}{k}$ till na representing new equilibrium position. It corresponds to increase of the spring elastic force energy from $\frac{k(n-1)^2q^2}{2}$ till $\frac{kn^2q^2}{2}$. As it is not hard to see corresponding energy difference can be expressed in the following way

$$\Delta E_{2n} = \frac{kn^2q^2}{2} - \frac{k(n-1)^2q^2}{2} \simeq knq^2 = knqq = nmgq \qquad \text{for} \quad n \gg 1. \tag{6}$$

It can be considered as the work of the gravitational force by moving of the mass nm for q. In this way increase of the energy of elastic force represents here direct consequence of the positive work of gravitational force.

Then, complete, positive, difference of the energy of elastic force from initial 0 value till final (after free falling of $\frac{N}{2}$ drops) value (2) can be simply obtained by formula

$$\Delta E_2 = (1 + 2 + ...\frac{N}{2})kq^2 = \frac{1}{2}\frac{N}{2}(1 + \frac{N}{2})kq^2 \simeq \frac{1}{4}\frac{1}{2}kN^2q^2 = \frac{1}{4}M^2\frac{g^2}{2k} = E_{fin2}. \tag{7}$$

It is very important to be pointed out this positive difference of the initial energy of elastic force is caused by positive work of gravitational force at the second box with discretely chengeable mass.

On the other side, after free falling of the first drop initially second box mass $M = Nm$ decreases for m till $(N-1)m$. For this reason new gravitational force $(N-1)mg$ becomes smaller than elastic force of the spring kNq. It causes compression of the spring, i.e. moving of the box up from Nq for $q = \frac{mg}{k}$ till $(N-1)q$ representing new equilibrium position (we consider absolute value of the position!). It causes decrease of the spring elastic force energy from $\frac{1}{2}kN^2q^2$ till $\frac{1}{2}k(N-1)^2q^2 \simeq \frac{1}{2}kN^2q^2 - kNq^2$ for energy difference $-kNq^2 = -kNqq = -Nmgq$. As it is not hard to see given energy difference can be considered as the negative work of the gravitational force by moving of the mass Nm for q.

By simple induction we can conclude the following. After free falling of the n-th drop in the second box, mass of the first box decreases from $(N-(n-1))m$ for m till $(N-n)m$. For this reason

new gravitational force $(N-n)mg$ becomes smaller than elastic force of the spring $(N-(n-1))kq$. It causes further compression of the spring, i.e. moving of the box up from $(N-(n-1))q$ for $q=\frac{mg}{k}$ till $(N-n)q$ representing new equilibrium position (we consider absolute value of the position!). It causes decrease of the spring elastic force energy from $k(N-(n-1))^2\frac{q^2}{2}$ till $k(N-n)^2\frac{q^2}{2}$. As it is not hard to see corresponding energy difference can be expressed in the following way

$$\Delta E_{1n} = k(N-n)^2\frac{q^2}{2} - k(N-(n-1))^2\frac{q^2}{2} - knq^2 = -knqq = -nmgq \qquad \text{for} \quad n \gg 1. \tag{8}$$

It can be considered as the negative work of the gravitational force by moving of the mass nm for q. In this way decrease of the energy of elastic force represents here direct consequence of the negative work of gravitational force.

Then, complete, negative, difference of the energy of elastic force from initial value $\frac{1}{2}kN^2q^2$ till final (after free falling of $\frac{N}{2}$ drops) value (2) can be simply obtained by formula

$$\Delta E_1 = -(N+(N-1)+...(N/2-1))kq^2 = \tag{9}$$

$$-\frac{1}{2}\frac{N}{2}(1+3\frac{N}{2})kq^2 \simeq -(\frac{3}{4})\frac{1}{2}kN^2q^2 = -(\frac{3}{4})M^2\frac{g^2}{2k} = -(\frac{3}{4})E_{in}$$

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It is very important to be pointed out this negative difference of the initial energy of elastic force is caused by negative work of gravitational force at the first box with discretely chengeable mass.

Total difference of the total energy of both elastic forces from initial value (3) till final value (5) equals, according to (7) and (9),

$$\Delta E = \Delta E_1 + \Delta E_2 == -\frac{3}{4}E_{in} + \frac{1}{4}E_{in} = -\frac{E_{in}}{2}. \tag{10}$$

In this way we obtain very simple and reasonable solution of two box paradox. Simply speaking "loss", i.e. negative total difference of the total elastic energy of both systems is result of the total negative work of the gravitational force by moving of the systems.

Finally, it can be observed that all this can be formulated completely analogously in the though (gedanken) experiment form by changing of the liquid drops by photons, i.e. by use of the pair of original Einstein-Bohr devices. (Photon, of course, cannot arise from the first in the other box by free fall. However, photon can arise from the first in the other box by an appropriate mirror.) It represents an interesting quantum analog of two capacitor paradox, but it goes over basic intentions of given work.

In conclusion, the following can be shortly repeated and pointed out. As it is well-known one of the most fascinating examples in remarkable discussion between Einstein and Bohr on the conceptual foundation of the quantum mechanics (Heisenberg energy-time uncertainty relation especially) was an experimental device representing a box hanged on an elastic spring. The pair of similar devices is used in this work for formulation of a classical and (implicitly) quantum mechanical analog of the famous two capacitor paradox. It admits a simple solution of the paradox since energy difference or seeming paradoxical "loss" can be explained by work of the elastic force for moving of the boxes in the gravitational field.(Obviously, original two capacitors paradox can be explained in the analogous way.)

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